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A refined theory for thick spherical shells

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Abstract

A theory for thick spherical shells is presented in this paper. The equations given here do not only incorporate the effects of transverse shear deformation but also account for the initial curvature as well as the radial stress. The proposed theory presents a very good approximation for the shell constitutive equations and the nonlinear distribution of the in plane stresses across the thickness of the shell. The later is very important for thick shell analyses. The presented formulation is based on the following: (1) assumed out of plane stress components which satisfy given boundary conditions; (2) three-dimensional elasticity equations with an integral form of the equilibrium equations; (3) stress resultants and stress couples acting on the middle surface of the shell, average displacements along the normal at a point on the middle surface, and average rotations of the normal.

The proposed shell equations can be conveniently used in finite element analysis. An application of this theory to the finite element analysis of spherical shells will be presented in the follow-up paper.

Keywords: Theory of shells; Thick shells; Transverse Shear strains; Radial stresses

1. Introduction

The complete two-dimensional theory of thin shells was developed by Love over 100 years ago. Numerous contributions on this subject have been made since then. Any two-dimensional theory of shells is an approximation of the real three-dimensional problem. Researchers have been seeking better approximations for the exact three-dimensional elasticity solutions for shells. In the last three decades, the developed refined two-dimensional linear theories of thin shells include important contributions of Sanders (1959), Flugge (1960), and Niordson (1978). In these refined shell theories, the initial curvature effect is taken into consideration in the formulation of shell equations. Nevertheless, the deformation is based on the Kirchhoff–Love assumption, and the radial stress effect is neglected. In the current work we will refer to all the theories built on Kirchhoff–Love assumption, as *the classical theory*. The refined theories by Sanders (1959), Flugge (1960) and Niordson (1978) provide very good results for the analysis of thin shells. The theory of Sanders–Koiter has been widely used in

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the finite element analysis of shells (Ashwell and Gallagher, 1976). However it is shown (Niordson, 1971) that Love's strain energy expression has inherent errors of relative order $[h/R + (h/L)^2]$ where h is the thickness of the shell, R is the magnitude of the smallest principal radius of curvature, and L is a characteristic wavelength of the deformation pattern of the middle surface. Consequently when the refined theories of thin shells are applied to thick shells, with h/R not small compared to unity, the error could be quite large as expected. Relative to theory of thin shells, the theory of thick shells has received limited attention by the researchers up to now. With the increase of the utilization of thick shells to various engineering applications such as cooling towers, dams, pressure vessels, etc. it is imperative to develop a simple and accurate theory for thick shells. Voyiadjis and Shi (1991) developed very accurate and convenient for finite element analysis a refined shell theory for thick cylindrical shells. The current work presents a refined shell theory for thick spherical shells, however the shell equations derived here are based on the same assumptions as those of Voyiadjis and Shi (1991). The proposed work can be considered a more general formulation of the Voyiadjis and Shi theory (1991).

Thick shells have a number of distinctly different features from thin shells. One of these features is that in thick shells the transverse shear deformation may no longer be neglected. In a number of particular cases of loadings the radial stress distribution of thick shells is very important and needs to be incorporated in the shell analysis. A third important distinction between thick and thin shell analyses is that in thick shell analysis the initial curvatures do not only contribute to the stress resultants and stress couples, but also result in nonlinear distribution of the in-plane stresses across the thickness of the shell.

It is not difficult to incorporate transverse shear deformations in shells. This can be accomplished following the work of Reissner (1945) for the plate theory. Nevertheless, it is not an easy task to incorporate radial stresses in thin shell theory and to obtain nonlinear stress distributions through the shell thickness in order to describe the behavior of thick shells. The attention in the previously developed shell theories was focused on the two-dimensional shell equations together with maintaining a linear stress distribution through the shell thickness (Flugge, 1960; Niordson, 1985). It appears that refinement of the stress distribution in thick shells has not received much attention with respect to the inclusion of radial stresses. The theory of thin shells may provide a good estimate of the strain energy for some problems in thick shells. Nevertheless, it cannot provide an accurate distribution of the stresses through the thickness (Gupta and Khatua, 1978). This accuracy is imperative from an engineering point of view.

The formulation procedure for the proposed shell theory is based on the following:

1. assumed out of plane stress components that satisfy given traction boundary conditions;
2. three-dimensional elasticity equations with an integral form of the equilibrium equations;
3. stress resultants and stress couples acting on the middle surface of the shell together with average displacements along a normal of the middle surface of the shell and the average rotations of the normal (Voyiadjis and Baluch, 1981).

It is well established that curved beams exhibit a nonlinear circumferential stress distribution through the thickness. In the proposed shell theory, all the in-plane stresses exhibit a nonlinear distribution through the thickness. This is primarily due to the incorporation of the initial curvature effect in the theoretical formulation of the proposed shell theory. The nonlinear stress expressions given here are compared for specific examples to those obtained through the three-dimensional theory of elasticity.

The resulting constitutive equations of shells reduce to those given by Flugge (1960) when the shear deformation and radial effects are neglected. In this case the average displacement is replaced by the middle surface displacements. However, the resulting equations are slightly different from those given by Sanders (1959), Koiter (1960) and Niordson (1978). This is primarily because the so-called effective stress tensor and effective moment tensor are used in the derivation of the constitutive equations instead of the usual stress tensors (Niordson, 1971).

The proposed shell equations can be conveniently used in the finite element analysis.

2. Theoretical formulation of the refined theory of thick spherical shells

2.1. Displacement field

The following out-of-plane stress components are assumed:

$$\sigma_z = \frac{(r_2/r)^3 - 1}{c_1} p_{\mathrm{i}} + \frac{(r_1/r)^3 - 1}{c_2} p_{\mathrm{o}} \tag{1}$$

where

$$c_1 = 1 - \left(\frac{r_2}{r_1}\right)^3 \tag{2}$$

$$c_2 = \left(\frac{r_1}{r_2}\right)^3 - 1 \tag{3}$$

$$\tau_{\theta z} = \left(1 + \frac{z}{R}\right) \frac{3Q_\theta}{2h} \left[1 - \left(\frac{2z}{h}\right)^2\right] + \frac{(r_2/r)^3 - 1}{c_1} p_{\theta \mathrm{i}} + \frac{(r_1/r)^3 - 1}{c_2} p_{\theta \mathrm{o}} \tag{4}$$

$$\tau_{\phi z} = \left(1 + \frac{z}{R}\right) \frac{3Q_\phi}{2h} \left[1 - \left(\frac{2z}{h}\right)^2\right] + \frac{(r_2/r)^3 - 1}{c_1} p_{\phi \mathrm{i}} + \frac{(r_1/r)^3 - 1}{c_2} p_{\phi \mathrm{o}} \tag{5}$$

$$r = R + z \tag{6}$$

where

σ_z radial stresses
$\tau_{\theta z}$, $\tau_{\phi z}$ transverse shear stresses (first subscripts—θ and ϕ denote the direction of the normal to the plane on which stresses are acting; second subscripts—z denote the direction of the stresses)
p_{i}, p_{o} distributed radial loads on the inner and outer surfaces respectively ($z = -h/2$ and $z = h/2$)
$p_{\theta \mathrm{i}}$, $p_{\theta \mathrm{o}}$ distributed loads along the θ direction, on the inner and outer surfaces respectively
$p_{\phi \mathrm{i}}$, $p_{\phi \mathrm{o}}$ distributed loads along the ϕ direction
r_1, r_2 radius of curvature of the inner and outer surface respectively (Fig. 1)
R radius of curvature of the mid-plane (Fig. 1)
Q_θ, Q_ϕ transverse shear forces
h thickness of the shell

Expression (1) depicts the radial stress distribution obtained from the elasticity solution for thick spheres subjected to constant radial loads at both surfaces $z = -h/2$ and $z = h/2$. The normal stress σ_z is ignored in the analysis of thin shells. Eqs. (4) and (5) express the transverse shear stress as obtained for a rectangular cross-section, modified by the term $(1 + z/R)$, due to the fact that the cross-section is not rectangular but exhibits a curvature. We notice that the modification applied here is different than the one most commonly used, i.e. $(1 - z/R)$, see Ugural (1981). This is due to different orientation of z axis which points outwards here. We therefore apply the usual modification term $(1 - z/R)$, and change the sign of z which is negative below the mid-section, obtaining $(1 + z/R)$. The assumed stress field (Eqs. (1)–(5)), satisfies the following boundary conditions:

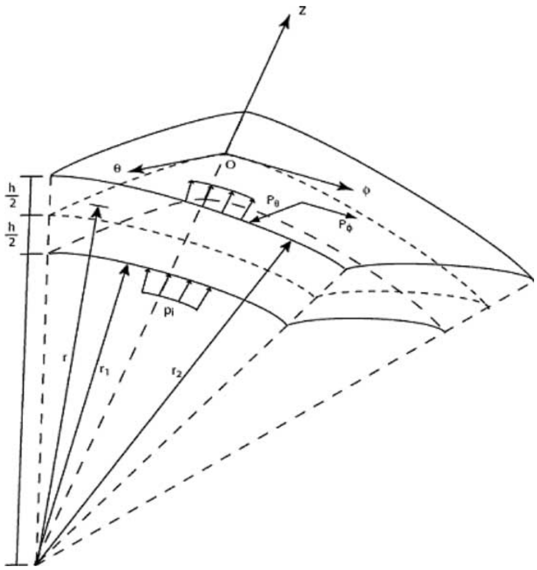

Fig. 1. Spherical shell element.

$$\begin{aligned} \sigma_z &= p_o && \text{at } z = h/2 \\ \tau_{z\phi} &= p_{\phi o} && \text{at } z = h/2 \\ \tau_{z\theta} &= p_{\theta o} && \text{at } z = h/2 \\ \sigma_z &= -p_i && \text{at } z = -h/2 \\ \tau_{z\phi} &= -p_{\phi i} && \text{at } z = -h/2 \\ \tau_{z\theta} &= -p_{\theta i} && \text{at } z = -h/2 \end{aligned} \tag{7}$$

Using Hooke's law for a linear elastic material, we obtain the transverse normal strain ε_z in terms of the stresses as follows:

$$\varepsilon_z = \frac{1}{E}[\sigma_z - v(\sigma_\theta + \sigma_\phi)] \tag{8}$$

The sum of $(\sigma_\theta + \sigma_\phi)$ can be written as indicated below:

$$\sigma_\theta + \sigma_\phi = \frac{12(M_\theta + M_\phi)z}{h^3} \tag{9}$$

Eq. (8) was first used by Reissner (1975) to modify the expression for the transverse displacement w. Substituting expressions (1) and (9) into Eq. (8), we obtain

$$\frac{\partial w}{\partial z} = \frac{1}{E}\left[\frac{(r_2/r)^3 - 1}{c_1}p_i + \frac{(r_1/r)^3 - 1}{c_2}p_o - \frac{12v}{h^3}(M_\theta + M_\phi)z\right] \tag{10}$$

Integrating Eq. (10) with respect to z yields the following expression for the displacement w:

$$w(\theta,\phi,z) = w_0(\theta,\phi) + \frac{1}{E}\int\left[\frac{(r_2/r)^3 - 1}{c_1}p_i + \frac{(r_1/r)^3 - 1}{c_2}p_o - \frac{12v}{h^3}(M_\theta + M_\phi)z\right]\mathrm{d}z \tag{11}$$

Denoting

$$M = (M_\theta + M_\phi) \tag{12}$$

and representing $1/(R+z)$ as a power series:

$$\frac{1}{R+z} = \frac{1}{R} - \frac{z}{R^2} + \frac{z^2}{R^3} - \cdots \tag{13}$$

we have

$$w(\theta,\phi,z) = w_0(\theta,\phi) + \frac{1}{E}\left\{\frac{p_{\mathrm{i}}}{c_1}\left[-z + \frac{r_2^3}{R^3}\left(z - \frac{3}{2}\frac{z^2}{R}\right)\right] + \frac{p_{\mathrm{o}}}{c_2}\left[-z + \frac{r_1^3}{R^3}\left(z - \frac{3}{2}\frac{z^2}{R}\right)\right] - v\frac{6z^2}{h^3}M\right\} \tag{14}$$

In the classical theory of bending of thin shells, the term z/R and its higher-order terms are neglected. In the present formulation, the term z/R is retained but all its higher-order terms are neglected. Eq. (14) is the resulting expression for $w(\theta,\phi,z)$.

In order to obtain consistent assumptions for the displacements $u(\theta,\phi,z)$ and $v(\theta,\phi,z)$, the following strain–displacement relations are used:

$$\frac{\partial v}{\partial z} + \frac{1}{(R+z)}\frac{\partial w}{\partial \phi} - \frac{v}{(R+z)} = \gamma_{\phi z} = \frac{\tau_{\phi z}}{G} \tag{15}$$

$$\frac{1}{(R+z)\sin\phi}\frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial z} - \frac{u}{(R+z)} = \gamma_{\theta z} = \frac{\tau_{\theta z}}{G} \tag{16}$$

where u, v, w are the displacements along θ, ϕ, z axes respectively.

Substituting for the appropriate shearing stress from expressions (4) and (5) into Eqs. (15) and (16), and integrating both expressions with respect to z, we obtain the remaining components of the displacement field:

$$\begin{aligned} u(\theta,\phi,z) = (1+z/R)\Bigg\{ & u_0(\theta,\phi) + \frac{Q_\theta}{2Gh}z\left[3 - \frac{4z^2}{h^2}\right] - \frac{1}{R\sin\phi}\frac{\partial w_0}{\partial \theta}\left(z - \frac{z^2}{R}\right) \\ & + \frac{2v}{Eh^3}\frac{1}{R\sin\phi}\frac{\partial M}{\partial \theta}z^3\left(1 - \frac{3z}{2R}\right) - \frac{1}{Ec_1}\frac{1}{R\sin\phi}\frac{\partial p_{\mathrm{i}}}{\partial \theta}\left[-\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_2^3}{R^3}\left(\frac{z^2}{2} - \frac{7z^3}{6R}\right)\right] \\ & - \frac{1}{Ec_2}\frac{1}{R\sin\phi}\frac{\partial p_{\mathrm{o}}}{\partial \theta}\left[-\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_1^3}{R^3}\left(\frac{z^2}{2} - \frac{7z^3}{6R}\right)\right] + \frac{p_{\theta\mathrm{i}}}{Gc_1}\left[-z + \frac{z^2}{2R} + \frac{r_2^3}{R^3}\left(z - \frac{2z^2}{R}\right)\right] \\ & + \frac{p_{\theta\mathrm{o}}}{Gc_2}\left[-z + \frac{z^2}{2R} + \frac{r_1^3}{R^3}\left(z - \frac{2z^2}{R}\right)\right]\Bigg\} \end{aligned} \tag{17}$$

$$\begin{aligned} v(\theta,\phi,z) = (1+z/R)\Bigg\{ & v_0(\theta,\phi) + \frac{Q_\phi}{2Gh}z\left[3 - \frac{4z^2}{h^2}\right] - \frac{1}{R}\frac{\partial w_0}{\partial \phi}\left(z - \frac{z^2}{R}\right) + \frac{2v}{Eh^3}\frac{1}{R}\frac{\partial M}{\partial \phi}z^3\left(1 - \frac{3z}{2R}\right) \\ & - \frac{1}{Ec_1}\frac{1}{R}\frac{\partial p_{\mathrm{i}}}{\partial \phi}\left[-\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_2^3}{R^3}\left(\frac{z^2}{2} - \frac{7z^3}{6R}\right)\right] - \frac{1}{Ec_2}\frac{1}{R}\frac{\partial p_{\mathrm{o}}}{\partial \phi}\left[-\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_1^3}{R^3}\left(\frac{z^2}{2} - \frac{7z^3}{6R}\right)\right] \\ & + \frac{p_{\phi\mathrm{i}}}{Gc_1}\left[-z + \frac{z^2}{2R} + \frac{r_2^3}{R^3}\left(z - \frac{2z^2}{R}\right)\right] + \frac{p_{\phi o}}{Gc_2}\left[-z + \frac{z^2}{2R} + \frac{r_1^3}{R^3}\left(z - \frac{2z^2}{R}\right)\right]\Bigg\} \end{aligned} \tag{18}$$

In the shell theory that follows the variations of the distributed loads $p_{\phi\mathrm{i}}$, $p_{\phi\mathrm{o}}$, $p_{\theta\mathrm{i}}$, $p_{\theta\mathrm{o}}$ are omitted for simplicity and conciseness. The reader may choose to include them by following the procedure outlined below.

2.2. Stress components

In order to obtain the remaining stress components, the following three-dimensional stress–strain relations are used:

$$\sigma_\theta = \frac{E}{(1-\nu^2)}[\varepsilon_\theta + \nu\varepsilon_\phi] + \frac{\nu}{1-\nu}\sigma_z \tag{19}$$

$$\sigma_\phi = \frac{E}{(1-\nu^2)}[\varepsilon_\phi + \nu\varepsilon_\theta] + \frac{\nu}{1-\nu}\sigma_z \tag{20}$$

$$\tau_{\theta\phi} = G\gamma_{\theta\phi} \tag{21}$$

together with the following strain–displacement relations:

$$\varepsilon_\theta = \frac{1}{(R+z)\sin\phi}\frac{\partial u}{\partial\theta} + \frac{v}{(R+z)}\operatorname{ctg}\phi + \frac{w}{R+z} \tag{22}$$

$$\varepsilon_\phi = \frac{1}{(R+z)}\frac{\partial v}{\partial\phi} + \frac{w}{R+z} \tag{23}$$

$$\gamma_{\theta\phi} = \frac{1}{(R+z)\sin\phi}\frac{\partial v}{\partial\theta} + \frac{1}{(R+z)}\frac{\partial u}{\partial\phi} - \frac{u}{(R+z)}\operatorname{ctg}\phi \tag{24}$$

Substituting for the displacements u, v and w from Eqs. (14), (17) and (18) respectively, into expressions (22)–(24) and substituting the resulting strain expressions into Eqs. (19)–(21), we obtain the following expressions for the stresses:

$$\begin{aligned}
\sigma_\theta = {} & \frac{E}{1-\nu^2} + \left\{ \frac{1}{R\sin\phi}\frac{\partial u_0}{\partial\theta} + \frac{\cos\phi}{R\sin\phi}v_0 + \frac{\nu}{R}\frac{\partial v_0}{\partial\phi} + \frac{z}{2Gh}\left[3 - \frac{4z^2}{h^2}\right]\left[\frac{1}{R\sin\phi}\frac{\partial Q_\theta}{\partial\theta} + \frac{\cos\phi}{R\sin\phi}Q_\phi + \frac{\nu}{R}\frac{\partial Q_\phi}{\partial\phi}\right]\right. \\
& + \Delta_1^2\left[-\frac{1}{R}\left(z - \frac{z^2}{R}\right)w_0 + \frac{2\nu}{Eh^3}\frac{1}{R}z^3\left(1 - \frac{3z}{2R}\right)M - \frac{1}{Ec_1}\frac{1}{R}\left[-\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_2^3}{R^3}\left(\frac{z^2}{2} - \frac{7z^3}{6R}\right)\right]p_\mathrm{i}\right. \\
& \left. - \frac{1}{Ec_2}\frac{1}{R}\left[-\frac{z^2}{2} + \frac{2z^3}{3R} + \frac{r_1^3}{R^3}\left(\frac{z^2}{2} - \frac{7z^3}{6R}\right)\right]p_\mathrm{o}\right] + \frac{p_{\phi\mathrm{i}}\cos\phi}{Gc_1R\sin\phi}\left[-z + \frac{z^2}{2R} + \frac{r_2^3}{R^3}\left(z - \frac{2z^2}{R}\right)\right] \\
& + \frac{p_{\phi\mathrm{o}}\cos\phi}{Gc_2R\sin\phi}\left[-z + \frac{z^2}{2R} + \frac{r_1^3}{R^3}\left(z - \frac{2z^2}{R}\right)\right] + \frac{1+\nu}{R\left(1+\frac{z}{R}\right)}\left[w_0 + \frac{1}{E}\left\{\frac{p_\mathrm{i}}{c_1}\left[-z + \frac{r_2^3}{R^3}\left(z - \frac{3}{2}\frac{z^2}{R}\right)\right]\right.\right. \\
& \left.\left.\left. + \frac{p_\mathrm{o}}{c_2}\left[-z + \frac{r_1^3}{R^3}\left(z - \frac{3}{2}\frac{z^2}{R}\right)\right] - \nu\frac{6z^2}{h^3}M\right\}\right]\right\} + \frac{\nu}{1+\nu}\left[\frac{p_\mathrm{i}}{c_1}\left(\frac{r_2^3}{(R+z)^3} - 1\right) + \frac{p_\mathrm{o}}{c_2}\left(\frac{r_1^3}{(R+z)^3} - 1\right)\right]
\end{aligned} \tag{25}$$

where

$$\Delta_1^2 = \frac{1}{R\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\phi}{R\sin\phi}\frac{\partial}{\partial\phi} + \frac{\nu}{R}\frac{\partial^2}{\partial\phi^2} \tag{26}$$

$$
\begin{aligned}
\sigma_\phi = \frac{E}{1-v^2}\Bigg\{ & \frac{1}{R\sin\phi}\frac{\partial u_0}{\partial\theta} + \frac{v\cos\phi}{R\sin\phi}v_0 + \frac{1}{R}\frac{\partial v_0}{\partial\phi} + \frac{z}{2Gh}\left[3-\frac{4z^2}{h^2}\right]\left[\frac{v}{R\sin\phi}\frac{\partial Q_\theta}{\partial\theta} + \frac{v\cos\phi}{R\sin\phi}Q_\phi \right.\\
& \left. + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi}\right] + \varDelta_2^2\left[-\frac{1}{R}\left(z-\frac{z^2}{R}\right)w_0 + \frac{2v}{Eh^3}\frac{1}{R}z^3\left(1-\frac{3z}{2R}\right)M \right.\\
& - \frac{1}{Ec_1}\frac{1}{R}\left[-\frac{z^2}{2}+\frac{2z^3}{3R}+\frac{r_2^3}{R^3}\left(\frac{z^2}{2}-\frac{7z^3}{6R}\right)\right]p_{\mathrm{i}} - \frac{1}{Ec_2}\frac{1}{R}\left.\left[-\frac{z^2}{2}+\frac{2z^3}{3R}+\frac{r_1^3}{R^3}\left(\frac{z^2}{2}-\frac{7z^3}{6R}\right)\right]p_{\mathrm{o}}\right]\\
& + \frac{vp_{\phi\mathrm{i}}\cos\phi}{Gc_1R\sin\phi}\left[-z+\frac{z^2}{2R}+\frac{r_2^3}{R^3}\left(z-\frac{2z^2}{R}\right)\right] + \frac{vp_{\phi\mathrm{o}}\cos\phi}{Gc_2R\sin\phi}\left[-z+\frac{z^2}{2R}+\frac{r_1^3}{R^3}\left(z-\frac{2z^2}{R}\right)\right]\\
& + \frac{1+v}{R\left(1+\frac{z}{R}\right)}\left[w_0+\frac{1}{E}\left\{\frac{p_{\mathrm{i}}}{c_1}\left[-z+\frac{r_2^3}{R^3}\left(z-\frac{3}{2}\frac{z^2}{R}\right)\right]+\frac{p_{\mathrm{o}}}{c_2}\left[-z+\frac{r_1^3}{R^3}\left(z-\frac{3}{2}\frac{z^2}{R}\right)\right]-v\frac{6z^2}{h^3}M\right\}\right]\Bigg\}\\
& + \frac{v}{1+v}\left[\frac{p_{\mathrm{i}}}{c_1}\left(\frac{r_2^3}{(R+z)^3}-1\right)+\frac{p_{\mathrm{o}}}{c_2}\left(\frac{r_1^3}{(R+z)^3}-1\right)\right]
\end{aligned}
\tag{27}
$$

where

$$
\varDelta_2^2 = \frac{v}{R\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{v\cos\phi}{R\sin\phi}\frac{\partial}{\partial\phi} + \frac{1}{R}\frac{\partial^2}{\partial\phi^2}
\tag{28}
$$

and

$$
\begin{aligned}
\tau_{\theta\phi} = G\Bigg\{ & \frac{1}{R\sin\phi}\frac{\partial v_0}{\partial\theta} + \frac{1}{R}\frac{\partial u_0}{\partial\phi} - \frac{u_0\cos\phi}{R\sin\phi}v_0 + \frac{z}{2Gh}\left[3-\frac{4z^2}{h^2}\right]\left[\frac{1}{R\sin\phi}\frac{\partial Q_\phi}{\partial\theta} + \frac{1}{R}\frac{\partial Q_\theta}{\partial\phi} - \frac{\cos\phi}{R\sin\phi}Q_\theta\right]\\
& + \varDelta_3^2\left[-\frac{1}{R}\left(z-\frac{z^2}{R}\right)w_0 + \frac{2v}{Eh^3}\frac{1}{R}z^3\left(1-\frac{3z}{2R}\right)M - \frac{1}{Ec_1}\frac{1}{R}\left[-\frac{z^2}{2}+\frac{2z^3}{3R}+\frac{r_2^3}{R^3}\left(\frac{z^2}{2}-\frac{7z^3}{6R}\right)\right]p_{\mathrm{i}} \right.\\
& \left. - \frac{1}{Ec_2}\frac{1}{R}\left[-\frac{z^2}{2}+\frac{2z^3}{3R}+\frac{r_1^3}{R^3}\left(\frac{z^2}{2}-\frac{7z^3}{6R}\right)\right]p_{\mathrm{o}}\right] + \frac{p_{\phi\mathrm{i}}\cos\phi}{Gc_1R\sin\phi}\left[-z+\frac{z^2}{2R}+\frac{r_2^3}{R^3}\left(z-\frac{2z^2}{R}\right)\right]\\
& + \frac{p_{\phi\mathrm{o}}\cos\phi}{Gc_2R\sin\phi}\left[-z+\frac{z^2}{2R}+\frac{r_1^3}{R^3}\left(z-\frac{2z^2}{R}\right)\right]\Bigg\}
\end{aligned}
\tag{29}
$$

where

$$
\varDelta_3^2 = \frac{2}{R\sin\phi}\frac{\partial^2}{\partial\theta\partial\phi} - \frac{2\cos\phi}{R\sin^2\phi}\frac{\partial}{\partial\theta}
\tag{30}
$$

2.3. Stress couples and stress resultants on the middle surface

Using the definitions of the stress couples:

$$
M_\theta = -\int_{-h/2}^{h/2}\sigma_\theta z\left(1+\frac{z}{R}\right)\mathrm{d}z
\tag{31}
$$

$$
M_\phi = -\int_{-h/2}^{h/2}\sigma_\phi z\left(1+\frac{z}{R}\right)\mathrm{d}z
\tag{32}
$$

$$M_{\theta\phi} = -\int_{-h/2}^{h/2} \tau_{\theta\phi} z \left(1 + \frac{z}{R}\right) \mathrm{d}z \tag{33}$$

We now substitute the expressions for stresses from Eqs. (25), (27), (29) into the respective relations for the stress couples to obtain

$$\begin{aligned} M_\theta = D\Bigg\{ & -\frac{1}{R^2 \sin\phi}\frac{\partial u_0}{\partial\theta} - \frac{\cos\phi}{R^2\sin\phi} v_0 - \frac{v}{R^2}\frac{\partial v_0}{\partial\phi} - \frac{6}{5Gh}\left[\frac{1}{R\sin\phi}\frac{\partial Q_\theta}{\partial\theta} + \frac{\cos\phi}{R\sin\phi}Q_\phi + \frac{v}{R}\frac{\partial Q_\phi}{\partial\phi}\right] \\ & + \frac{1}{R}\Delta_1^2 w_0 + \left(\frac{9vh}{112ER^3} - \frac{3v}{10ERh}\right)\Delta_1^2 M + \frac{1}{Ec_1}\frac{1}{R}\left[\frac{h^2}{40R}\left(1 - 4\frac{r_2^3}{R^3}\right)\right]\Delta_1^2 p_\mathrm{i} \\ & + \frac{1}{Ec_2}\frac{1}{R}\left[\frac{h^2}{40R}\left(1 - 4\frac{r_1^3}{R^3}\right)\right]\Delta_1^2 p_\mathrm{o} + \frac{p_{\phi\mathrm{i}}\cos\phi}{R\sin\phi}\frac{1}{Gc_1}\left[1 - \frac{r_2^3}{R^3} - \frac{3h^2}{40R^2}\right] \\ & + \frac{p_{\phi\mathrm{o}}\cos\phi}{R\sin\phi}\frac{1}{Gc_2}\left[1 - \frac{r_1^3}{R^3} - \frac{3h^2}{40R^2}\right] + \frac{1+v}{ER}\left[\frac{p_\mathrm{i}}{c_1}\left[1 + v - \frac{r_2^3}{R^3}(1-2v)\right]\right. \\ & \left.\left. + \frac{p_\mathrm{o}}{c_2}\left[1 + v - \frac{r_1^3}{R^3}(1-2v)\right]\right]\right\} \end{aligned} \tag{34}$$

$$\begin{aligned} M_\phi = D\Bigg\{ & -\frac{v}{R^2 \sin\phi}\frac{\partial u_0}{\partial\theta} - \frac{v\cos\phi}{R^2\sin\phi} v_0 - \frac{1}{R^2}\frac{\partial v_0}{\partial\phi} - \frac{6}{5Gh}\left[\frac{v}{R\sin\phi}\frac{\partial Q_\theta}{\partial\theta} + \frac{v\cos\phi}{R\sin\phi}Q_\phi + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi}\right] \\ & + \frac{1}{R}\Delta_2^2 w_0 + \left(\frac{9vh}{112ER^3} - \frac{3v}{10ERh}\right)\Delta_2^2 M + \frac{1}{Ec_1}\frac{1}{R}\left[\frac{h^2}{40R}\left(1 - 4\frac{r_2^3}{R^3}\right)\right]\Delta_2^2 p_\mathrm{i} \\ & + \frac{1}{Ec_2}\frac{1}{R}\left[\frac{h^2}{40R}\left(1 - 4\frac{r_1^3}{R^3}\right)\right]\Delta_2^2 p_\mathrm{o} + \frac{vp_{\phi\mathrm{i}}\cos\phi}{R\sin\phi}\frac{1}{Gc_1}\left[1 - \frac{r_2^3}{R^3} - \frac{3h^2}{40R^2}\right] \\ & + \frac{vp_{\phi\mathrm{o}}\cos\phi}{R\sin\phi}\frac{1}{Gc_2}\left[1 - \frac{r_1^3}{R^3} - \frac{3h^2}{40R^2}\right] + \frac{1+v}{ER}\left[\frac{p_\mathrm{i}}{c_1}\left[1 + v - \frac{r_2^3}{R^3}(1-2v)\right]\right. \\ & \left.\left. + \frac{p_\mathrm{o}}{c_2}\left[1 + v - \frac{r_1^3}{R^3}(1-2v)\right]\right]\right\} \end{aligned} \tag{35}$$

and

$$\begin{aligned} M_{\theta\phi} = D\frac{1-v}{2}\Bigg\{ & -\frac{1}{R^2 \sin\phi}\frac{\partial v_0}{\partial\theta} - \frac{1}{R^2}\frac{\partial u_0}{\partial\phi} - \frac{\cos\phi}{R^2\sin\phi} u_0 - \frac{6}{5Gh}\left[\frac{1}{R\sin\phi}\frac{\partial Q_\phi}{\partial\theta} + \frac{1}{R}\frac{\partial Q_\theta}{\partial\phi} - \frac{\cos\phi}{R\sin\phi}Q_\theta\right] \\ & + \frac{1}{R}\Delta_3^2 w_0 + \left(\frac{9vh}{112ER^3} - \frac{3v}{10ERh}\right)\Delta_3^2 M + \frac{1}{Ec_1}\frac{1}{R}\left[\frac{h^2}{40R}\left(1 - 4\frac{r_2^3}{R^3}\right)\right]\Delta_3^2 p_\mathrm{i} \\ & + \frac{1}{Ec_2}\frac{1}{R}\left[\frac{h^2}{40R}\left(1 - 4\frac{r_1^3}{R^3}\right)\right]\Delta_3^2 p_\mathrm{o} + \frac{p_{\theta\mathrm{i}}\cos\phi}{R\sin\phi}\frac{1}{Gc_1}\left[1 - \frac{r_2^3}{R^3} - \frac{3h^2}{40R^2}\right] \\ & + \frac{p_{\theta\mathrm{o}}\cos\phi}{R\sin\phi}\frac{1}{Gc_2}\left[1 - \frac{r_1^3}{R^3} - \frac{3h^2}{40R^2}\right]\Bigg\}. \end{aligned} \tag{36}$$

Substituting for the stresses σ_θ, σ_ϕ, and τ_ϕ from Eqs. (25), (27), (29) into the following definitions for the stress resultants:

$$N_\theta = \int_{-h/2}^{h/2} \sigma_\theta \left(1 + \frac{z}{R}\right) \mathrm{d}z \tag{37}$$

$$N_\phi = \int_{-h/2}^{h/2} \sigma_\phi \left(1 + \frac{z}{R}\right) \mathrm{d}z \tag{38}$$

$$N_{\theta\phi} = \int_{-h/2}^{h/2} \tau_{\theta\phi} \left(1 + \frac{z}{R}\right) \mathrm{d}z \tag{39}$$

we obtain the following expressions for the stress resultants:

$$\begin{aligned} N_\theta = \frac{Eh}{1-v^2} \Bigg\{ & \frac{1}{R\sin\phi}\frac{\partial u_0}{\partial\theta} + \frac{\cos\phi}{R\sin\phi} v_0 + \frac{v}{R}\frac{\partial v_0}{\partial\phi} + \frac{h}{10GR}\left[\frac{1}{R\sin\phi}\frac{\partial Q_\theta}{\partial\theta} + \frac{\cos\phi}{R\sin\phi} Q_\phi + \frac{v}{R}\frac{\partial Q_\phi}{\partial\phi}\right] \\ & - \frac{vh}{80ER^2}\Delta_1^2 M + \frac{1}{Ec_1}\left[\frac{h^2}{24R}\left(1 - \frac{r_2^3}{R^3}\right)\right]\Delta_1^2 p_\mathrm{i} + \frac{1}{Ec_2}\left[\frac{h^2}{24R}\left(1 - \frac{r_1^3}{R^3}\right)\right]\Delta_1^2 p_\mathrm{o} \\ & - \frac{p_{\phi\mathrm{i}}\cos\phi}{\sin\phi}\frac{1}{Gc_1}\left[\frac{h^2}{24R^2}\left(1 + 2\frac{r_2^3}{R^3}\right)\right] - \frac{p_{\phi\mathrm{o}}\cos\phi}{\sin\phi}\frac{1}{Gc_2}\left[\frac{h^2}{24R^2}\left(1 + 2\frac{r_1^3}{R^3}\right)\right] + \frac{1+v}{R} w_0 \\ & - \frac{3v(1+v)}{10EhR} M - \frac{v(1-v)}{E}\left[\frac{p_\mathrm{i}}{c_1}\left(1 - \frac{r_2^3}{R^3}\right) + \frac{p_\mathrm{o}}{c_2}\left(1 - \frac{r_1^3}{R^3}\right)\right]\Bigg\} \end{aligned} \tag{40}$$

$$\begin{aligned} N_\phi = \frac{Eh}{1-v^2} \Bigg\{ & \frac{v}{R\sin\phi}\frac{\partial u_0}{\partial\theta} + \frac{v\cos\phi}{R\sin\phi} v_0 + \frac{1}{R}\frac{\partial v_0}{\partial\phi} + \frac{h}{10GR}\left[\frac{v}{R\sin\phi}\frac{\partial Q_\theta}{\partial\theta} + \frac{v\cos\phi}{R\sin\phi} Q_\phi + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi}\right] \\ & - \frac{vh}{80ER^2}\Delta_2^2 M + \frac{1}{Ec_1}\left[\frac{h^2}{24R}\left(1 - \frac{r_2^3}{R^3}\right)\right]\Delta_2^2 p_\mathrm{i} + \frac{1}{Ec_2}\left[\frac{h^2}{24R}\left(1 - \frac{r_1^3}{R^3}\right)\right]\Delta_2^2 p_\mathrm{o} \\ & - \frac{p_{\phi i} v\cos\phi}{\sin\phi}\frac{1}{Gc_1}\left[\frac{h^2}{24R^2}\left(1 + 2\frac{r_2^3}{R^3}\right)\right] - \frac{p_{\phi\mathrm{o}} v\cos\phi}{\sin\phi}\frac{1}{Gc_2}\left[\frac{h^2}{24R^2}\left(1 + 2\frac{r_1^3}{R^3}\right)\right] + \frac{1+v}{R} w_0 \\ & - \frac{3v(1+v)}{10EhR} M - \frac{v(1-v)}{E}\left[\frac{p_\mathrm{i}}{c_1}\left(1 - \frac{r_2^3}{R^3}\right) + \frac{p_\mathrm{o}}{c_2}\left(1 - \frac{r_1^3}{R^3}\right)\right]\Bigg\} \end{aligned} \tag{41}$$

and

$$\begin{aligned} N_{\theta\phi} = \frac{Eh}{1-v^2}\left(\frac{1-v}{2}\right) \Bigg\{ & \frac{1}{R\sin\phi}\frac{\partial v_0}{\partial\theta} + \frac{1}{R}\frac{\partial u_0}{\partial\phi} - \frac{\cos\phi}{R\sin\phi} u_0 + \frac{h}{10GR}\left[\frac{1}{R\sin\phi}\frac{\partial Q_\phi}{\partial\theta} + \frac{1}{R}\frac{\partial Q_\theta}{\partial\phi} - \frac{\cos\phi}{R\sin\phi} Q_\theta\right] \\ & - \frac{vh}{80ER^2}\Delta_3^2 M + \frac{1}{Ec_1}\left[\frac{h^2}{24R}\left(1 - \frac{r_2^3}{R^3}\right)\right]\Delta_3^2 p_\mathrm{i} + \frac{1}{Ec_2}\left[\frac{h^2}{24R}\left(1 - \frac{r_1^3}{R^3}\right)\right]\Delta_3^2 p_\mathrm{o} \\ & - \frac{p_{\phi\mathrm{i}}\cos\phi}{\sin\phi}\frac{1}{Gc_1}\left[\frac{h^2}{24R^2}\left(1 + 2\frac{r_2^3}{R^3}\right)\right] - \frac{p_{\phi\mathrm{o}}\cos\phi}{\sin\phi}\frac{1}{Gc_2}\left[\frac{h^2}{24R^2}\left(1 + 2\frac{r_1^3}{R^3}\right)\right]\Bigg\}. \end{aligned} \tag{42}$$

2.4. Average displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ and rotations ϕ_θ, ϕ_ϕ

For identifying the proper boundary conditions of the derived shell theory, average displacements $\bar{u}$, $\bar{v}$, $\bar{w}$, and rotations ϕ_θ, ϕ_ϕ are introduced. The rotations are for sections $\theta = \text{const}$ and $\phi = \text{const}$, respectively. We first define transverse shear resultants as

$$Q_\theta = T\gamma_{\theta z} \tag{43}$$

$$Q_\phi = T\gamma_{\phi z} \tag{44}$$

where T is given by

$$T = \frac{5}{6}Gh \tag{45}$$

and $\gamma_{\theta z}$, $\gamma_{\phi z}$ expressed similarly to Eqs. (15) and (16):

$$\gamma_{\theta z} = \frac{1}{(R+z)\sin\phi}\frac{\partial \bar{w}}{\partial \theta} + \frac{\partial \bar{u}}{\partial z} - \frac{\bar{u}}{(R+z)} \tag{46}$$

$$\gamma_{\phi z} = \frac{\partial \bar{v}}{\partial z} + \frac{1}{(R+z)}\frac{\partial \bar{w}}{\partial \phi} - \frac{\bar{v}}{(R+z)} \tag{47}$$

The average transverse displacement $\bar{w}$ is obtained by equating the work of the transverse shear stress $\tau_{\phi z}$ due to the displacement w to the work of the transverse shear resultant Q_ϕ due to the average displacement $\bar{w}$ (Voyiadjis and Baluch, 1981):

$$\int_{-h/2}^{h/2} \tau_{\phi z} w\left(1+\frac{z}{R}\right)\mathrm{d}z = Q_\phi \bar{w} \tag{48}$$

One could choose to equate the work of the transverse shear stress $\tau_{\theta z}$ due to the displacement w to the work of the transverse shear resultant Q_θ due to the average displacement $\bar{w}$ instead, which yields the same resulting expression for $\bar{w}$, given by

$$\bar{w} = w_0 - M\left(\frac{3v}{10Eh} - \frac{9vh}{112ER^2}\right) - \frac{1}{10}\frac{h^2}{REc_1}\frac{r_2^3}{R^3}p_\mathrm{i} - \frac{1}{10}\frac{h^2}{REc_2}\frac{r_1^3}{R^3}p_\mathrm{o} \tag{49}$$

Similarly to obtain $\bar{u}$, $\bar{v}$, ϕ_θ, ϕ_ϕ we use the following equations:

$$\int_{-h/2}^{h/2} \sigma_\theta u\left(1+\frac{z}{R}\right)\mathrm{d}z = N_\theta \bar{u} + M_\theta \phi_\theta \tag{50}$$

$$\int_{-h/2}^{h/2} \sigma_\phi v\left(1+\frac{z}{R}\right)\mathrm{d}z = N_\phi \bar{v} + M_\phi \phi_\phi \tag{51}$$

The resulting expressions for $\bar{u}$, $\bar{v}$, ϕ_θ, ϕ_ϕ are given by

$$\bar{u} = u_0 + \frac{1}{ER\sin\phi}\frac{h^2}{24}\left[\frac{1}{c_1}\frac{\partial p_\mathrm{i}}{\partial \theta}\left(1-\frac{r_2^3}{R^3}\right) + \frac{1}{c_2}\frac{\partial p_\mathrm{o}}{\partial \theta}\left(1-\frac{r_1^3}{R^3}\right)\right] \tag{52}$$

$$\bar{v} = v_0 + \frac{1}{ER}\frac{h^2}{24}\left[\frac{1}{c_1}\frac{\partial p_\mathrm{i}}{\partial \phi}\left(1-\frac{r_2^3}{R^3}\right) + \frac{1}{c_2}\frac{\partial p_\mathrm{o}}{\partial \phi}\left(1-\frac{r_1^3}{R^3}\right)\right] \tag{53}$$

$$\phi_\theta = \frac{1}{R\sin\phi}\frac{\partial \bar{w}}{\partial \theta} - \frac{6}{5Gh}Q_\theta \tag{54}$$

$$\phi_\phi = \frac{1}{R}\frac{\partial \bar{w}}{\partial \phi} - \frac{6}{5Gh}Q_\phi \tag{55}$$

Making use of Eqs. (43) and (44) we can write the following:

$$\phi_\theta = \frac{1}{R\sin\phi}\frac{\partial \bar{w}}{\partial \theta} - \gamma_{\theta z} \tag{56}$$

$$\phi_\phi = \frac{1}{R}\frac{\partial \bar{w}}{\partial \phi} - \gamma_{\phi z}. \tag{57}$$

The remaining stress resultants and stress couples can be expressed in a more concise manner in terms of $\bar{u}$, $\bar{v}$, $\bar{w}$, $\gamma_{\theta z}$, $\gamma_{\phi z}$ as follows:

$$M_\theta = D\left[\frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{R\sin\phi}\frac{\partial\bar{w}}{\partial\theta} - \gamma_{\theta z}\right) + \frac{1}{R}\operatorname{ctg}\phi\left(\frac{1}{R}\frac{\partial\bar{w}}{\partial\phi} - \gamma_{\phi z}\right) + \frac{v}{R}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\frac{\partial\bar{w}}{\partial\phi} - \gamma_{\phi z}\right)\right.$$
$$\left. - \frac{1}{R^2\sin\phi}\frac{\partial\bar{u}}{\partial\theta} - \frac{\operatorname{ctg}\phi}{R^2}\bar{v} - \frac{v}{R^2}\frac{\partial\bar{v}}{\partial\phi}\right] + k_1 p_{\mathrm{i}} + k_2 p_{\mathrm{o}} + k_3 p_{\phi\mathrm{i}} + k_4 p_{\phi\mathrm{o}} \tag{58}$$

$$M_\phi = D\left[\frac{v}{R\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{R\sin\phi}\frac{\partial\bar{w}}{\partial\theta} - \gamma_{\theta z}\right) + \frac{v}{R}\operatorname{ctg}\phi\left(\frac{1}{R}\frac{\partial\bar{w}}{\partial\phi} - \gamma_{\phi z}\right) + \frac{1}{R}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\frac{\partial\bar{w}}{\partial\phi} - \gamma_{\phi z}\right)\right.$$
$$\left. - \frac{v}{R^2\sin\phi}\frac{\partial\bar{u}}{\partial\theta} - \frac{v\operatorname{ctg}\phi}{R^2}\bar{v} - \frac{1}{R^2}\frac{\partial\bar{v}}{\partial\phi}\right] + k_1 p_{\mathrm{i}} + k_2 p_{\mathrm{o}} + v k_3 p_{\phi\mathrm{i}} + v k_4 p_{\phi\mathrm{o}} \tag{59}$$

$$M_{\theta\phi} = D\frac{1-v}{2}\left[\frac{1}{R}\frac{\partial}{\partial\phi}\left(\frac{1}{R\sin\phi}\frac{\partial\bar{w}}{\partial\theta} - \gamma_{\theta z}\right) + \frac{1}{R}\operatorname{ctg}\phi\left(\frac{1}{R\sin\phi}\frac{\partial\bar{w}}{\partial\theta} - \gamma_{\phi z}\right) + \frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\frac{\partial\bar{w}}{\partial\phi} - \gamma_{\phi z}\right)\right.$$
$$\left. + \frac{\operatorname{ctg}\phi}{R}\left(\frac{1}{R\sin\phi}\frac{\partial\bar{w}}{\partial\theta}\right) - \frac{1}{R^2\sin\phi}\frac{\partial\bar{v}}{\partial\theta} - \frac{1}{R^2}\frac{\partial\bar{u}}{\partial\phi} + \frac{\operatorname{ctg}\phi}{R^2}\bar{u}\right] + k_5 p_{\mathrm{i}} + k_6 p_{\mathrm{o}} + k_3\frac{1-v}{2}p_{\phi\mathrm{i}} + k_4\frac{1-v}{2}p_{\phi\mathrm{o}} \tag{60}$$

$$N_\theta = \frac{Eh}{1-v^2}\left[\frac{1}{R\sin\phi}\frac{\partial\bar{u}}{\partial\theta} + \frac{1}{R}\operatorname{ctg}\phi\bar{v} + \frac{v}{R}\frac{\partial\bar{v}}{\partial\phi} + \frac{1+v}{R}\bar{w}\right] + D\left[\frac{1}{R^2\sin\phi}\frac{\partial\gamma_{\theta z}}{\partial\theta} + \frac{\operatorname{ctg}\phi}{R^2}\gamma_{\phi z} + \frac{v}{R^2}\frac{\partial\gamma_{\phi z}}{\partial\phi}\right]$$
$$+ k_7 p_{\mathrm{i}} + k_8 p_{\mathrm{o}} + k_9 p_{\phi\mathrm{i}} + k_{10} p_{\phi\mathrm{o}} \tag{61}$$

$$N_\phi = \frac{Eh}{1-v^2}\left[\frac{v}{R\sin\phi}\frac{\partial\bar{u}}{\partial\theta} + \frac{v}{R}\operatorname{ctg}\phi\bar{v} + \frac{1}{R}\frac{\partial\bar{v}}{\partial\phi} + \frac{1+v}{R}\bar{w}\right] + D\left[\frac{v}{R^2\sin\phi}\frac{\partial\gamma_{\theta z}}{\partial\theta} + \frac{v\operatorname{ctg}\phi}{R^2}\gamma_{\phi z} + \frac{1}{R^2}\frac{\partial\gamma_{\phi z}}{\partial\phi}\right]$$
$$+ k_7 p_{\mathrm{i}} + k_8 p_{\mathrm{o}} + v k_9 p_{\phi\mathrm{i}} + v k_{10} p_{\phi\mathrm{o}} \tag{62}$$

$$N_{\theta\phi} = \frac{Eh}{1-v^2}\left(\frac{1-v}{2}\right)\left[\frac{1}{R\sin\phi}\frac{\partial\bar{v}}{\partial\theta} + \frac{1}{R}\frac{\partial\bar{u}}{\partial\phi} - \frac{1}{R}\operatorname{ctg}\phi\bar{u}\right] + D\left(\frac{1-v}{2}\right)\left[\frac{1}{R^2\sin\phi}\frac{\partial\gamma_{\phi z}}{\partial\theta} - \frac{\operatorname{ctg}\phi}{R^2}\gamma_{\theta z} + \frac{1}{R^2}\frac{\partial\gamma_{\theta z}}{\partial\phi}\right]$$
$$- \left(\frac{1-v}{2}\right)k_9 p_{\theta i} + \left(\frac{1-v}{2}\right)k_{10} p_{\theta\mathrm{o}} \tag{63}$$

where

$$k_1 = D\frac{1+v}{ERc_1}\left[1 + v - \frac{r_2^3}{R^3}(1-2v)\right] \tag{64}$$

$$k_2 = D\frac{1+v}{ERc_2}\left[1 + v - \frac{r_1^3}{R^3}(1-2v)\right] \tag{65}$$

$$k_3 = D\frac{\mathrm{ctg}\,\phi}{RGc_1}\left(1 - \frac{r_2^3}{R^3}\right) \tag{66}$$

$$k_4 = D\frac{\mathrm{ctg}\,\phi}{RGc_2}\left(1 - \frac{r_1^3}{R^3}\right) \tag{67}$$

$$k_5 = D\frac{1-v}{2}\frac{1}{ERc_1}\left[\frac{h^2}{40R}\left(1 - 4\frac{r_2^3}{R^3}\right)\right] \tag{68}$$

$$k_5 = D\frac{1-v}{2}\frac{1}{ERc_2}\left[\frac{h^2}{40R}\left(1 - 4\frac{r_1^3}{R^3}\right)\right] \tag{69}$$

$$k_7 = \frac{Eh}{1-v^2}\left[\frac{h^2}{10REc_1}\frac{r_2^3}{R^3}\frac{1+v}{R} - \frac{v(1+v)}{Ec_1}\left(1 - \frac{r_2^3}{R^3}\right)\right] \tag{70}$$

$$k_8 = \frac{Eh}{1-v^2}\left[\frac{h^2}{10REc_2}\frac{r_1^3}{R^3}\frac{1+v}{R} - \frac{v(1+v)}{Ec_2}\left(1 - \frac{r_1^3}{R^3}\right)\right] \tag{71}$$

$$k_9 = -\frac{Eh}{1-v^2}\left[\frac{\mathrm{ctg}\,\phi}{Gc_1}\left[\frac{h^2}{24R^2}\left(1 + 2\frac{r_2^3}{R^3}\right)\right]\right] \tag{72}$$

$$k_{10} = -\frac{Eh}{1-v^2}\left[\frac{\mathrm{ctg}\,\phi}{Gc_2}\left[\frac{h^2}{24R^2}\left(1 + 2\frac{r_1^3}{R^3}\right)\right]\right] \tag{73}$$

These resulting constitutive equations reduce to those given by Flugge (1960) when the shear deformation and radial effects are neglected. In this case, the average displacements are replaced by the middle surface displacements. The transverse shear forces Q_θ, Q_ϕ are obtained in this case from the equilibrium equations in terms of the stress couples.

An alternate set of expressions for the stress couples may be obtained in terms of the average displacements $\bar{u}$, $\bar{v}$, $\bar{w}$, and corresponding rotations ϕ_θ, ϕ_ϕ. The following relations give these equations:

$$M_\theta = D\left[\frac{1}{R\sin\phi}\frac{\partial\phi_\theta}{\partial\theta} + \frac{1}{R}\phi_\phi\mathrm{ctg}\,\phi + \frac{v}{R}\frac{\partial\phi_\phi}{\partial\phi} - \frac{1}{R^2\sin\phi}\frac{\partial\bar{u}}{\partial\theta} - \frac{\mathrm{ctg}\,\phi}{R^2}\bar{v} - \frac{v}{R^2}\frac{\partial\bar{v}}{\partial\phi}\right] + k_1p_\mathrm{i} + k_2p_\mathrm{o} + k_3p_{\phi\mathrm{i}} + k_4p_{\phi\mathrm{o}} \tag{74}$$

$$M_\phi = D\left[\frac{v}{R\sin\phi}\frac{\partial\phi_\theta}{\partial\theta} + \frac{v}{R}\phi_\phi\mathrm{ctg}\,\phi + \frac{1}{R}\frac{\partial\phi_\phi}{\partial\phi} - \frac{v}{R^2\sin\phi}\frac{\partial\bar{u}}{\partial\theta} - \frac{v\mathrm{ctg}\,\phi}{R^2}\bar{v} - \frac{1}{R^2}\frac{\partial\bar{v}}{\partial\phi}\right] + k_1p_\mathrm{i} + k_2p_\mathrm{o} + vk_3p_{\phi\mathrm{i}} + vk_4p_{\phi\mathrm{o}} \tag{75}$$

$$M_{\theta\phi} = D\frac{1-v}{2}\left[\frac{1}{R}\frac{\partial\phi_\theta}{\partial\phi} + \frac{1}{R}\phi_\theta\mathrm{ctg}\,\phi + \frac{1}{R\sin\phi}\frac{\partial\phi_\phi}{\partial\theta} + \frac{\mathrm{ctg}\,\phi}{R}(\phi_\theta + \gamma_\theta) - \frac{1}{R^2\sin\phi}\frac{\partial\bar{v}}{\partial\theta} - \frac{1}{R^2}\frac{\partial\bar{u}}{\partial\phi} + \frac{\mathrm{ctg}\,\phi}{R^2}\bar{u}\right] + k_5p_\mathrm{i} + k_6p_\mathrm{o} + k_3\frac{1-v}{2}p_{\phi\mathrm{i}} + k_4\frac{1-v}{2}p_{\phi\mathrm{o}} \tag{76}$$

2.5. Equilibrium equations and boundary conditions

A free body diagram is used to derive the equilibrium equations. For the case of small deformation analysis, the shell equilibrium equations are given by (Flugge, 1960):

$$\frac{\partial}{\partial \phi}(N_\phi \sin\phi) + \frac{\partial N_{\theta\phi}}{\partial \theta} - N_\theta \cos\phi - Q_\phi \sin\phi + R\sin\phi p_\phi = 0 \tag{77}$$

$$\frac{\partial}{\partial \phi}(N_{\theta\phi} \sin\phi) + \frac{\partial N_\theta}{\partial \theta} + N_{\theta\phi} \cos\phi - Q_\theta \sin\phi + R\sin\phi p_\theta = 0 \tag{78}$$

$$N_\theta \sin\phi + N_\phi \sin\phi + \frac{\partial Q_\theta}{\partial \theta} + \frac{\partial}{\partial \phi}(Q_\phi \sin\phi) - R\sin\phi p_z = 0 \tag{79}$$

$$\frac{\partial}{\partial \phi}(M_\phi \sin\phi) + \frac{\partial M_{\theta\phi}}{\partial \theta} - M_\theta \cos\phi - RQ_\phi \sin\phi = 0 \tag{80}$$

$$\frac{\partial}{\partial \phi}(M_{\theta\phi} \sin\phi) + \frac{\partial M_\theta}{\partial \theta} + M_{\theta\phi} \cos\phi - RQ_\theta \sin\phi = 0 \tag{81}$$

$$\frac{M_{\phi\theta}}{R} - \frac{M_{\theta\phi}}{R} = N_{\phi\theta} - N_{\theta\phi} \tag{82}$$

In the above equilibrium expressions, p_ϕ, p_θ, p_z are the equivalent distributed loads acting on the middle surface of the shell. Eq. (82) is identically satisfied consequently reducing the number of equilibrium equations to 5. The stress resultants and couples may be expressed in terms of either $\bar{u}$, $\bar{v}$, $\bar{w}$, γ_θ, γ_ϕ or $\bar{u}$, $\bar{v}$, $\bar{w}$, ϕ_θ, ϕ_ϕ. We therefore have five unknowns to solve for from the five remaining equilibrium conditions (77)–(81).

The static and kinematic boundary conditions may be expressed in terms of either $\bar{u}$, $\bar{v}$, $\bar{w}$, γ_θ, γ_ϕ or $\bar{u}$, $\bar{v}$, $\bar{w}$, ϕ_θ, ϕ_ϕ, together with the use of the constitutive equations (58)–(73). The boundary conditions are given as follows:

1. if edge $(0,\phi)$ is simply supported the BC's may be written as

   $\bar{w}(0,\phi) = 0; \quad \phi_\phi(0,\phi) = 0; \quad M_\theta(0,\phi) = 0$

2. if edge $(0,\phi)$ is clamped the BCs may be written as

   $\bar{w}(0,\phi) = 0; \quad \phi_\phi(0,\phi) = 0; \quad \phi_\theta(0,\phi) = 0; \quad \bar{u}(0,\phi) = 0$

3. if on the edge $(0,\phi)$ stretching of the mid-plane is prevented, BCs may be written as $u_0(0,\phi) = 0$; $v_0(0,\phi) = 0$, and if additionally the pressures p_z are uniformly distributed, i.e. $\frac{\partial p_z}{\partial \theta} = \frac{\partial p_z}{\partial \phi} = 0$ then $\bar{u}(0,\phi) = 0$; $\bar{v}(0,\phi) = 0$
4. if edge $(0,\phi)$ is free to stretch in θ direction, then $v_0(0,\phi) = 0$; $N_\theta(0,\phi) = 0$
5. if edge $(0,\phi)$ is free the BCs may be written as

   $M_\theta(0,\phi) = 0; \quad Q_\theta(0,\phi) = 0; \quad M_{\theta\phi}(0,\phi) = 0; \quad N_\theta(0,\phi) = 0; \quad N_{\theta\phi}(0,\phi) = 0.$

2.6. The nonlinear nature of the stress distribution

The resulting nonlinear distribution through the thickness for the in-plane stresses in the proposed thick shell theory is due to the incorporation of the initial curvature of the shell, and the three-dimensional constitutive equations as obtained from relations (19)–(21). This effect becomes highly pronounced in thick shells by changing the magnitude of the maximum stress significantly as compared to the linear stress variation theory.

In the expressions for in-plane stress components σ_θ, σ_ϕ, $\tau_{\theta\phi}$ given by Eqs. (25)–(29), nonlinear terms such as $1/(R+z)$ and z^2/R are incorporated. Consequently, the stresses given by the present theory have a nonlinear distribution along the thickness of the shell. Let us consider the simple case of a constant normal pressure and investigate the corresponding stress distribution of σ_ϕ through the thickness. In this case we have

$$\begin{aligned}\sigma_\phi = \frac{E}{1-\nu^2}\Bigg\{ & \frac{1}{R\sin\phi}\frac{\partial u_0}{\partial\theta} + \frac{\nu\cos\phi}{R\sin\phi}v_0 + \frac{1}{R}\frac{\partial v_0}{\partial\phi} + \frac{z}{2Gh}\left[3 - \frac{4z^2}{h^2} - \frac{3z}{R}\left(1 - \frac{2z}{3R} - \frac{2z^2}{h^2}\right)\right] \\ & \times \left[\frac{\nu}{R\sin\phi}\frac{\partial Q_\theta}{\partial\theta} + \frac{\nu\cos\phi}{R\sin\phi}Q_\phi + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi}\right] + \Delta_2^2\left[-\frac{1}{R}\left(z - \frac{z^2}{R}\right)w_0 + \frac{2\nu}{Eh^3}\frac{1}{R}z^3\left(1 - \frac{3z}{2R}\right)M\right] \\ & + \frac{\nu p_{\phi i}\cos\phi}{Gc_1R\sin\phi}\left[-z + \frac{z^2}{2R} + \frac{r_2^3}{R^3}\left(z - \frac{2z^2}{R}\right)\right] + \frac{\nu p_{\phi o}\cos\phi}{Gc_2R\sin\phi}\left[-z + \frac{z^2}{2R} + \frac{r_1^3}{R^3}\left(z - \frac{2z^2}{R}\right)\right] \\ & + \frac{1+\nu}{R\left(1+\frac{z}{R}\right)}\left[w_0 + \frac{1}{E}\left\{\frac{p_i}{c_1}\left[-z + \frac{r_2^3}{R^3}\left(z - \frac{3}{2}\frac{z^2}{R}\right)\right] + \frac{p_o}{c_2}\left[-z + \frac{r_1^3}{R^3}\left(z - \frac{3}{2}\frac{z^2}{R}\right)\right] - \nu\frac{6z^2}{h^3}M\right\}\right]\Bigg\} \\ & + \frac{\nu}{1+\nu}\left[\frac{p_i}{c_1}\left(\frac{r_2^3}{(R+z)^3} - 1\right) + \frac{p_o}{c_2}\left(\frac{r_1^3}{(R+z)^3} - 1\right)\right] \end{aligned} \tag{83}$$

In Eq. (83) all the terms are nonlinear in z except for the terms associated with $\frac{1}{R\sin\phi}\frac{\partial u_0}{\partial\theta}, \frac{\partial v_0}{\partial\phi}, \frac{\partial^2 w_0}{\partial\theta^2}$.

The stress distribution obtained using the presented theory will be compared with the elasticity theory.

3. Equivalent formulation for the thick plate theory

It is relatively simple to reduce the proposed shell theory to a thick plate theory. As R approaches infinity the stress resultants and stress couples reduce to

$$N_x = \frac{Eh}{1-\nu^2}\left(\frac{\partial\bar{u}}{\partial x} + \nu\frac{\partial\bar{v}}{\partial y}\right) + k_1(p_i + p_o) \tag{84}$$

$$N_y = \frac{Eh}{1-\nu^2}\left(\frac{\partial\bar{v}}{\partial y} + \nu\frac{\partial\bar{u}}{\partial x}\right) + k_1(p_i + p_o) \tag{85}$$

$$N_x = N_y = Gh\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right) \tag{86}$$

$$Q_x = T\left(\frac{\partial\bar{w}}{\partial x} - \phi_x\right) \tag{87}$$

$$Q_y = T\left(\frac{\partial\bar{w}}{\partial y} - \phi_y\right) \tag{88}$$

$$M_x = D\left(\frac{\partial\phi_x}{\partial x} + v\frac{\partial\phi_y}{\partial y}\right) + k_2(p_i + p_o) \tag{89}$$

$$M_y = D\left(\frac{\partial\phi_y}{\partial y} + v\frac{\partial\phi_x}{\partial x}\right) + k_2(p_i + p_o) \tag{90}$$

$$M_{xy} = M_{yx} = D\frac{1-v}{2}\left(\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial x}\right) \tag{91}$$

where

$$k_1 = \frac{vh}{2(1-v)} \tag{92}$$

$$k_2 = -D\frac{6v(1+v)}{5Eh} \tag{93}$$

We note that the present shell theory reduces to exactly the same equivalent thick plate theory as the one given by Voyiadjis and Shi (1991).

4. Examples

4.1. Thick sphere subjected to uniform pressures

We investigate the stress distribution of σ_ϕ for a thick sphere subjected to uniform pressure $p_i = 5$ kPa, and $p_o = 4$ kPa (Fig. 2).

In this case we have

$$v = Q_\phi = \frac{\partial M_\phi}{\partial\phi} = 0 \tag{94}$$

and

$$w = w(z) \tag{95}$$

The stress σ_ϕ using the proposed theory is expressed in this case as follows:

$$\sigma_\phi = \frac{E}{R+z}\left\{w_0 + \frac{p_i}{c_1}\left[-z + \frac{r_2^3}{R^3}\left(z - \frac{3}{2}\frac{z^2}{R}\right)\right] + \frac{p_o}{c_2}\left[-z + \frac{r_1^3}{R^3}\left(z - \frac{3}{2}\frac{z^2}{R}\right)\right]\right\} \tag{96}$$

The corresponding exact elasticity solution for this problem is given by Lame (1833):

$$\sigma_\phi = -\frac{p_o}{2c_2}\left(2 + \frac{r_1^3}{(R+z)^3}\right) - \frac{p_i}{2c_1}\left(2 + \frac{r_2^3}{(R+z)^3}\right) \tag{97}$$

From the theory of elasticity we have

$$w_0 = \frac{R}{E}\sigma_\phi|_{z=0} \tag{98}$$

where

$$\sigma_\phi|_{z=0} = -\frac{p_o}{2c_2}\left(2 + \frac{r_1^3}{R^3}\right) - \frac{p_i}{2c_1}\left(2 + \frac{r_2^3}{R^3}\right) \tag{99}$$

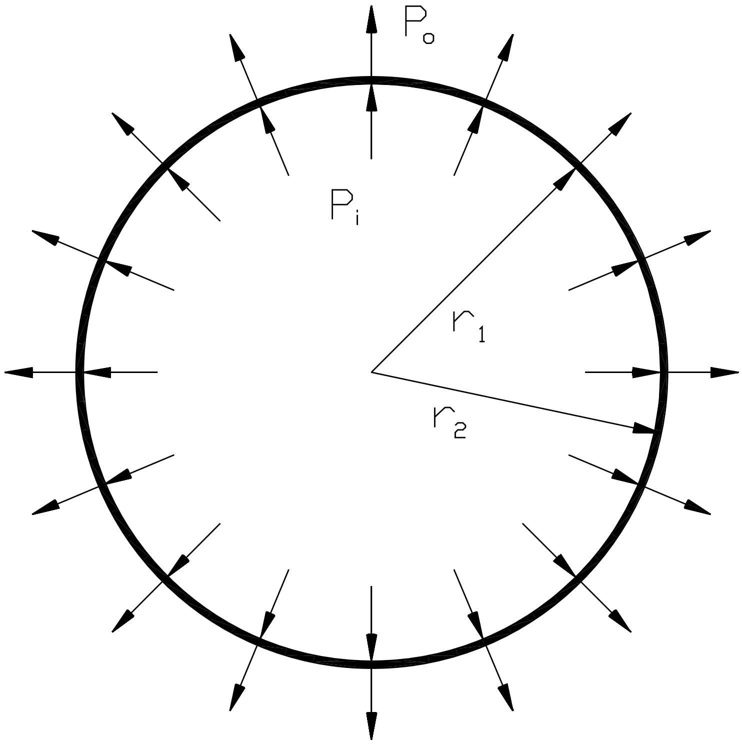

Fig. 2. Spherical shell subject to internal and external pressures.

Substituting for w_0 from Eqs. (98) and (99) into expression (96), we obtain the following expression for σ_ϕ:

$$\sigma_\phi = \frac{R}{R+z}\left[-\frac{p_o}{2c_2}\left(2+\frac{r_1^3}{R^3}\right)-\frac{p_i}{2c_1}\left(2+\frac{r_2^3}{R^3}\right)\right]+\frac{1}{R+z}\left[\frac{p_i}{Ec_1}\left(-z+\frac{r_2^3}{R^3}\left(z-\frac{3}{2}\frac{z^2}{R}\right)\right)\right.$$
$$\left.+\frac{p_o}{c_2}\left(-z+\frac{r_1^3}{R^3}\left(z-\frac{3}{2}\frac{z^2}{R}\right)\right)\right] \tag{100}$$

It can be easily shown that σ_ϕ obtained from Eq. (100) for the case of $z = 0$ is identical to σ_ϕ obtained from the elasticity solution expressed by Eq. (97), for the same case, i.e. $z = 0$ (Table 1). It is also worthy to mention that, as expected in the case of a sphere $\sigma_\phi = \sigma_\theta$.

Gupta and Khatua (1978) in their derivation of a thick shell superparametric finite element proposed a modification in the expression for the circumferential stress σ_ϕ. Their modified expression is given by

$$\sigma_\phi = \frac{R}{R+z}\sigma_0 \tag{101}$$

where σ_0 is the average hoop stress. We note that Gupta and Khatua's scheme cannot distinguish the difference between the internal and external pressures.

As shown in Table 1, the present theory is very close to the exact elasticity solution. In order to show the improvement in the present theory versus the classical shell theory, the problem of spherical container subject to uniform internal pressure $p_i = 5$ kPa is analyzed. Fig. 3 shows comparison of the exact solution with classical theory by Niordson (1985), and the present theory.

Table 1
σ_ϕ distribution for spherical shell

r_1	r_2	r_2/r_1	$h = r_2 - r_1$	c_1	c_2	Elasticity σ_ϕ (kPa)		Present theory σ_ϕ (kPa)	
						$r = r_1$	$r = r_2$	$r = r_1$	$r = r_2$
3	3.9	1.3	0.9	−1.2	−0.545	19.7782	15.2782	19.712	15.315
3	4.5	1.5	1.5	−2.4	−0.704	14.18421	9.68421	14.01	9.7539
3	5.1	1.7	2.1	−3.9	−0.796	11.95004	7.45004	11.633	7.5458
3	6	2	3	−7	−0.875	10.42857	5.92857	9.8571	6.0476
3	6.6	2.2	3.6	−9.6	−0.906	9.899254	5.39925	9.1463	5.5253

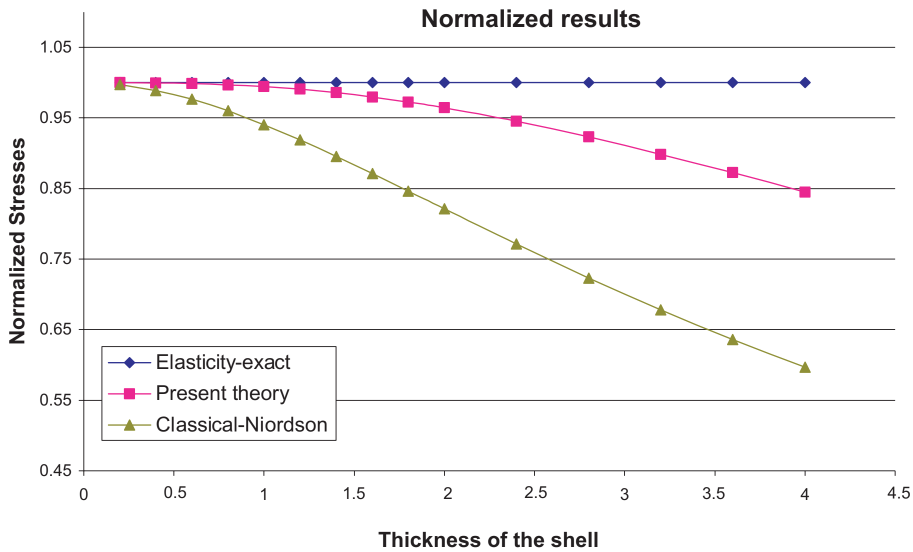

Fig. 3. Normalized σ_ϕ for spherical container subject to internal pressure.

As expected the results deviate form the exact, as the thickness of the shell increases. However, there is a significant improvement in the results obtained using the present theory when compared to the classical shell theory, which yields large errors for thick shells.

The error in the present work proves to be much smaller than in the case of classical thin shell theory. The later is built on Kirchhoff–Love assumption, which as shown by Niordson (1971) has relative error of $[h/R + (h/L)^2]$. We therefore expect the error of the classical theory to be very close to the expression given by Niordson: $[h/R + (h/L)^2]$. Comparison of errors is shown in Fig. 4.

The classical theory has an error that is approximately equal to the Niordson error. The present theory also shows some loss of accuracy as the thickness of the shell increases. It is however much smaller than the Niordson error, as shown in Fig. 4.

4.2. Hemispherical dome under uniform gravitational pressure

We consider a simply supported hemispherical dome of radius $R = 10$ m and thickness t, subject to gravitational pressure $p = 0.5$ kPa (Fig. 5).

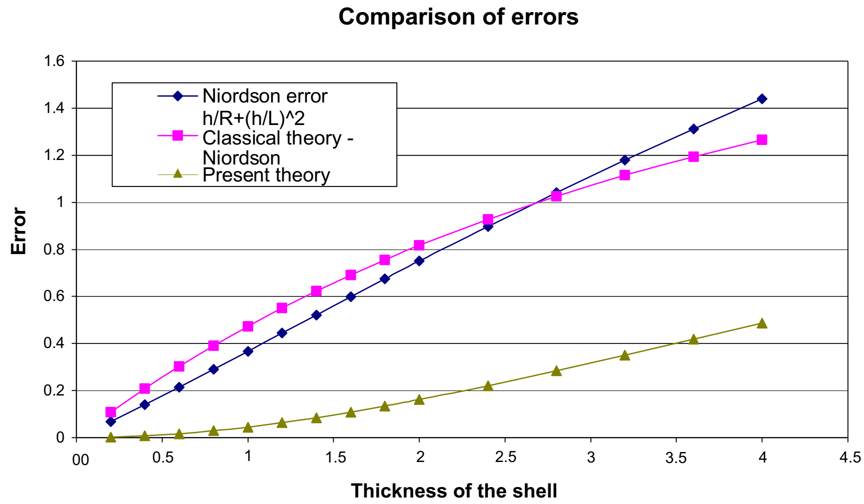

Fig. 4. Relative errors.

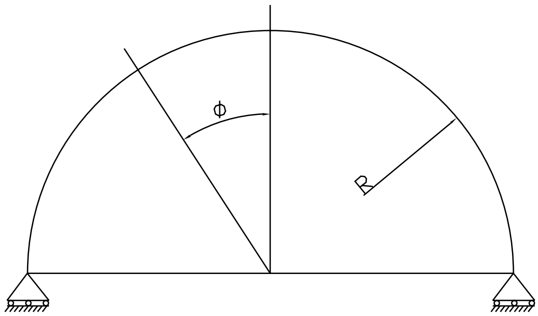

Fig. 5. Hemispherical dome.

The bending stresses reach maximum at the top of the dome, i.e. $\phi = 0°$. If the shell is thin, they are however considered negligible and the loading is entirely resisted by the membrane action of the shell. As the thickness increases, bending stresses with nonlinear terms start to play an important role.

We will investigate σ_θ stresses at $\phi = 0°$, i.e. at the top of the dome. The results of the analysis given by the classical theory and the present are shown in Table 2 and Fig. 6.

Analysis of the above results leads to the same conclusions as in the previous example. The present theory shows very good agreement with the classical one for the case of thin shells, while there is an improvement in the treatment of thick shells.

4.3. Morley's spherical shell

The following example is used as a standard problem to test the accuracy of the shell theories and the finite elements built based on these theories (MacNeal and Harder, 1985). The problem represents a hemisphere with four point loads alternating in sign at 90° intervals on the equator, which is a free edge (see Fig. 7).

Table 2
σ_θ distribution for spherical dome

Thickness, t (m)	Classical-Niordson σ_θ (kPa)	Present theory σ_θ (kPa)
0.06	−25.8065	−25.13
0.1	−15.7143	−13.93
0.14	−11.3553	−9.409
0.18	−8.91473	−6.965
0.22	−7.3501	−5.661
0.26	−6.25943	−4.906
0.3	−5.45455	−4.375
0.4	−4.13462	−3.226
0.6	−2.79412	−1.925
1.0	−1.7	−0.85

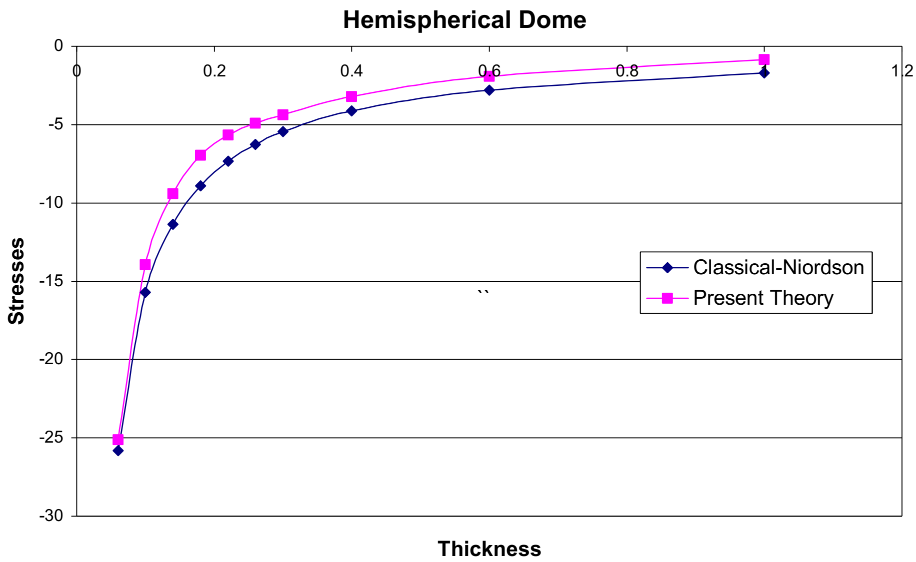

Fig. 6. Comparison of results.

Both membrane and bending strains contribute significantly to the radial displacement at the point of load application. The value of the displacement, 0.094 under the load, published by MacNeal and Harder (1985), is used as a reference solution. Steele (1987) and Simo et al. (1989) stated however that the analytical solution based on the asymptotic expansion yields an answer of 0.093. The present theory yields the value of **0.0929**, which once again proves the current work to be accurate.

We also investigate the transverse shear stresses for the problem above with different thicknesses for the shell. We compare the values obtained here with those by Mindlin/Koiter–Sanders theory in Table 3.

The normal stresses σ_x are shown here to compare the magnitudes of normal and transverse shear stresses. The last column in Table 3 gives the ratio of τ_{xz}/σ_x. It shows the increasing importance of the transverse shear stresses with the increase of the thickness of the shell. For the first shell analyzed, with a thickness of 0.04 in, τ_{xz} is only 0.0068 of the normal stresses σ_x, whereas the same ratio for the thickness of 0.9 in becomes 0.12. It shows the expected pattern of the transverse shear stresses becoming more significant in the case of thick shells.

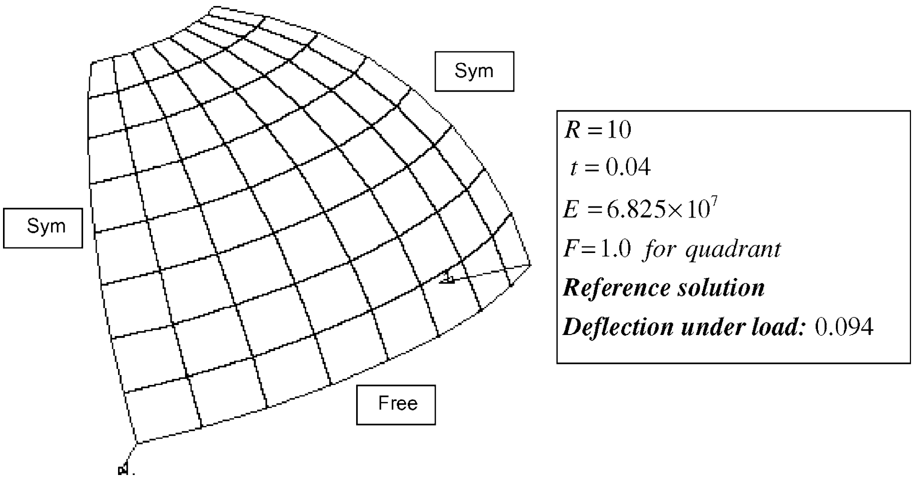

Fig. 7. Morley's sphere.

Table 3
Transverse shear and normal stresses for hemispherical shell

Thickness, t (in.)	τ_{xz} (psi)		τ_{yz} (psi)		σ_x (psi)		Ratio τ_{xz}/σ_x
	Mindlin/KS	Present	Mindlin/KS	Present	Mindlin/KS	Present	
0.04	−38.71	−38.5	−22.38	−22.21	−5691	−5658	0.0068
0.1	−15.11	−14.98	−6.7	−6.62	−965.7	−954.6	0.0156
0.18	−8.131	−7.96	−3.596	−3.51	−305.4	−298.303	0.0266
0.28	−5.047	−4.97	−2.417	−2.37	−127.3	−125.6	0.0396
0.4	−3.41	−3.19	−1.804	−1.703	−62.34	−61.63	0.0547
0.54	−2.441	−2.3	−1.42	−1.33	−33.93	−33.26	0.0719
0.7	−1.824	−1.642	−1.152	−1.121	−19.92	−19.67	0.0915
0.9	−1.376	−1.27	−0.9376	−0.926	−11.82	−11.68	0.1164

The present theory provides very good approximation of the transverse stresses which of particularly great importance in the case of thick shells.

4.4. Circular arch

Another benchmark problem testing the accuracy of the shell theories is cantilevered circular arch subject to in-plane shear (MacNeal and Harder, 1985). One end of the arch is fixed against displacements and rotations, and the other end is free. Inner radius = 4.12, outer radius = 4.32, thickness = 0.1, Young's modulus = 10E6, Poisson's ratio = 0.25. Two unit forces are applied at the free end of the arch (Fig. 8). Vertical deflection of the free end is investigated here. The analytical solution of this problem stated by MacNeal and Harder (1985) is 0.08734. The deflection resulting from the present theory yields the value of **0.08074**, which for the problem above is a very good approximation of the exact solution.

4.5. Thick cylinder subjected to uniform pressures

The current theory can be reduced to the case of cylindrical shells, as given by Voyiadjis and Shi (1991). Therefore, the Voyiadjis and Shi (1991) formulation can be regarded as a special case of the present theory.

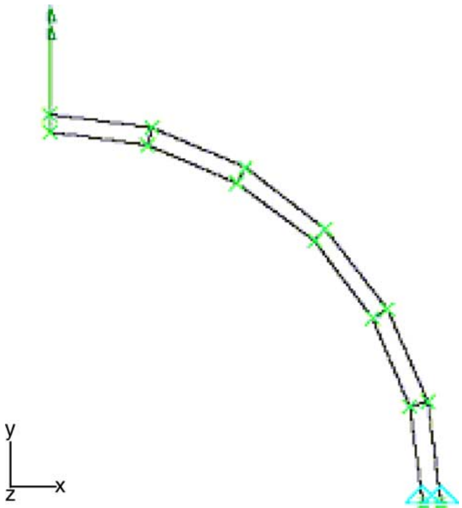

Fig. 8. Circular arch.

To show this we now investigate the stress distribution of σ_ϕ for a thick cylinder subjected to uniform pressures p_i and p_o. Similarly to the previous example, we have

$$v = Q_\phi = \frac{\partial M_\phi}{\partial \phi} = 0 \tag{102}$$

and

$$w = w(z) \tag{103}$$

To reduce the current theory to the case of cylindrical shells we adopt

$$R \sin\phi \partial\theta = \partial x \quad \text{and} \quad \frac{u}{R} = \frac{v}{R} = \frac{w}{R} = 0 \tag{104}$$

Considering also solutions due to Lame for thick cylinders, we can obtain the stress distribution for σ_ϕ as given by Voyiadjis and Shi for the case of cylindrical shells:

$$\sigma_\phi = \frac{E}{R+z}\left\{ w_0 + \frac{p_i}{Ec_1}\left[-z + \frac{r_2^2}{R^2}\left(z - \frac{z^2}{R} \right) \right] + \frac{p_o}{c_2}\left[-z + \frac{r_1^2}{R^2}\left(z - \frac{3}{2}\frac{z^2}{R} \right) \right] \right\} \tag{105}$$

The corresponding exact elasticity solution for this problem is given by

$$\sigma_\phi = -\frac{p_o}{c_2}\left(1 + \frac{r_1^2}{(R+z)^2} \right) - \frac{p_i}{c_1}\left(1 + \frac{r_2^2}{(R+z)^2} \right) \tag{106}$$

Table 4 shows comparison of the results of the given problem obtained by various theories with both analytical and numerical results obtained here. The numerical solution shown in Table 4, is obtained with doubly curved finite elements built on the present spherical theory. It shows very good agreement with the analytical solution of the cylindrical shell problem, provided by the same theory, as well as the exact-elasticity solution obtained by Lame (1833). It shows applicability of the present theory to not only spherical shells but also shells with different radius of curvature in two directions. The present theory can therefore be applied to shells of general shapes.

Table 4
σ_ϕ distribution for cylindrical shell

r_2/r_1	Winkler's theory		Elasticity-exact		Present theory			
					Analytical		Numerical	
	$r = r_1$	$r = r_2$	$r = r_1$	$r = r_2$	$r = r_1$	$r = r_2$	$r = r_1$	$r = r_2$
1.5	−26.971	20.607	−27.858	21.275	−27.971	20.029	−27.692	19.826
2	−7.725	4.863	−7.755	4.917	−7.642	4.358	−7.464	4.284
3	−2.285	1.095	−2.292	1.130	−2.105	0.925	−2.029	0.876

5. Conclusions

A theory for thick spherical shells is developed in this paper. By considering the shear strains, the transverse shear deformations are accounted for in the resulting shell equations. In the proposed theory, the initial curvature effect is incorporated in the stress distribution leading to an accurate nonlinear distribution of the in-plane stresses. Through the incorporation of the radial stresses to the proposed shell formulation, we obtain the stress resultants and stress couples associated not only with the middle surface displacement of the shells, but also with the radial stresses explicitly. By using the constitutive equations of the three-dimensional theory of elasticity and incorporating the initial curvature effect on the stress resultants and couples, an accurate set of constitutive equations for the thick shell theory is obtained.

The constitutive equations presented here reduce to those given by Flugge (1960) when the shear deformations and the radial stress effects are neglected, while the average displacements are replaced by the middle surface displacements of the shell. The resulting proposed equations in this paper are slightly different than those given by Sanders (1959), Koiter (1960) and Niordson (1978), primarily because they use the so-called effective stress resultants and stress couple tensors. These effective stresses are used in the variational derivation of the constitutive equations (see Niordson, 1985). However, even when both the shear deformation and the radial stresses are neglected, the stress distributions given in the present paper will still be nonlinear because the stresses are derived from the three-dimensional constitutive equations given by expressions (19)–(21).

The nonlinear distribution of the in-plane stresses through the thickness for thick shells was ignored in the past in the formulation of the theory. This is not the case in the present paper. The nonlinear distribution constitutes a very important ingredient for an accurate and reliable thick shell theory.

Similar to the shell theory by Sanders–Koiter, presented shell equations are convenient for use in the finite element analysis. The proposed theory is not only very useful in the analysis of thick shells, but also has the potential for use in the analysis of composite shells (see Noor and Burton, 1989). This theory is also important in applications of vibrations of shells where the shear deformation and stress distribution along the thickness direction play an important role.

The examples given here show that the proposed theory is accurate and in good agreement with the exact solution, and other existing theories. The classical theory of shells yields errors that could grow large in the case of moderate to thick shells. In the present theory there is a significant reduction in error, which is much smaller than in the case of the classical theory, based on the Kirchhoff–Love assumption. This is clearly shown in the first example. The current work is applicable to plates (setting the radius of curvature infinite), beams as special cases of plates, and through the use of the finite element method to shells of arbitrary shape, with radius of curvature being different in two directions e.g. cylindrical shells as well as arches. It is therefore general and universal and gives very good results for all of the above-discussed cases.

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References

Ashwell, D.G., Gallagher, R.H. (Eds.), 1976. Finite Elements for Thin Shells and Curved Membranes. Wiley, New York.

Flugge, W., 1960. Stresses in Shells. Springer, New York.

Gupta, A.K., Khatua, T.P., 1978. On thick superparametric shell element. Int. J. Numer. Methods Engng. 12, 1883–1889.

Koiter, W.T., 1960. A consistent first approximation in general theory of thin elastic shells. In: Koiter, W.T. (Ed.), Theory of Thin Elastic Shells, First IUTAM Symposium. North-Holland, Amsterdam, pp. 12–33.

Lame, G., 1833. Theory of Elasticity.

MacNeal, H.R., Harder, R.L., 1985. A proposed standard set of problems to test finite element accuracy. Finite Elements Anal. Design 1, 3–20.

Niordson, F.I., 1971. A note on the strain energy of elastic shells. Int. J. Solids Struct. 7, 1573–1579.

Niordson, F.I., 1978. A consistent refined shell theory. In: Complete Analysis and its Applications, Vekua Anniversary Volume. Nauka, Moscow, pp. 421–429.

Niordson, F.I., 1985. Shell Theory. North-Holland, Amsterdam.

Noor, A.K., Burton, W.S., 1989. Assessment of shear deformation theory for multilayered composite plates. Appl. Mech. Rev. 42 (1), 1–13.

Reissner, E., 1945. The effects of transverse shear deformation on the bending of elastic plates. J. Appl. Mech. ASME 12, 66–77.

Reissner, E., 1975. On transverse bending of plates, including the effects of transverse shear deformation. Int. J. Solids Struct. 11, 569–576.

Sanders, J.L., 1959. An improved first approximation theory of thin shells. NASA Report 24.

Simo, J.C., Fox, D.D., Rifai, M.S., 1989. On a stress resultant geometrically exact shell model. Comput. Methods Appl. Mech. Engng. 73, 53–92.

Steele, C.R., 1987. Private Communications.

Ugural, A.C., 1981. Stresses in Plates and Shells. McGraw-Hill.

Voyiadjis, G.Z., Baluch, H.M., 1981. Refined theory for flexural motions of isotropic plates. J. Sound Vib. 76, 57–64.

Voyiadjis, G.Z., Shi, G., 1991. Refined two-dimensional theory for thick cylindrical shells. Int. J. Solids Struct. 27, 261–282.